Spanish Relativity Meeting (ERE 2009) IOP Publishing
Journal of Physics: Conference Series **229** (2010) 012038 doi:10.1088/1742-6596/229/1/012038

Spinor calculus on 5-dimensional spacetimes

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Abstract. We explain how Penrose's spinor calculus of 4-dimensional Lorentzian geometry is implemented in a 5-dimensional Lorentzian manifold. A number of issues, such as the essential spin algebra, the spin covariant derivative and the algebro-differential properties of the curvature spinors are discussed .

1. Introduction

Recently, the study of General Relativity in dimensions higher than four has experienced much progress. Specific fields which have been the subject of interest are the discovery and classification of new exact solutions [3], the generalisation of the Petrov classification [1] and the Newman-Penrose formalism [6, 9] or the study of isolated bodies. When the spacetime dimension is five one may expect that the study of these subjects become simpler than when one works in generic dimension N. Indeed, there are important discoveries which took place first when the spacetime dimension was assumed to be five. Perhaps the best known example is the discovery of the *black ring* solution [2] which led to the realisation that the stationary black hole uniqueness theorems known in four dimensions are not generalisable to higher dimensions. In this work we contribute to this trend by showing how the spinor calculus of Penrose [7] which was developed on a four-dimensional spacetime is formulated in a spacetime of five dimensions.

2. Spinor algebra

Let $\mathbf{L}$ be a 5-dimensional real vector space endowed with a real scalar product $g(\ ,\ )$ of Lorentzian signature (signature convention $(+,-,-,-,-)$) and let $\mathbf{S}$ be a complex vector space whose dimension is for the moment left unspecified (complex conjugate of scalars will be denoted by an overbar). Using the vector space $\mathbf{L}$ and its dual $\mathbf{L}^*$ as the starting point one builds a tensor algebra in the standard fashion. Similarly a tensor algebra is built from $\mathbf{S}$ and its dual $\mathbf{S}^*$. We denote these algebras by $\mathfrak{T}(\mathbf{L})$ and $\mathfrak{T}(\mathbf{S})$ respectively [1]. In this work abstract indices will be used throughout to denote tensorial quantities: in this way small Latin indices $a, b, \dots$ will denote abstract indices on elements of $\mathfrak{T}(\mathbf{L})$ and capital Latin indices $A, B, \dots$ will be used for abstract

[1] Strictly speaking only the algebras $\mathfrak{T}^r_s(\mathbf{L})$ of tensors r-contravariant s-covariant can be defined (and the same applies to $\mathfrak{T}^r_s(\mathbf{S})$). To lessen the notation we will suppress the labels r, s in the notation and they will only be made explicit when confusion may arise.

indices of elements in $\mathfrak{T}(\mathbf{S})$. The tensor algebra $\mathfrak{T}(\mathbf{S})$ will be referred to as the *spin algebra* and its elements will be called spinors. One can also build tensor algebras by taking tensor products of elements in $\mathfrak{T}(\mathbf{L})$ and elements in $\mathfrak{T}(\mathbf{S})$. Quantities in these tensor algebras will be referred to as *mixed tensors* and they will carry abstract indices of both types. The algebras $\mathfrak{T}(\mathbf{L})$ and $\mathfrak{T}(\mathbf{S})$ shall be regarded as complex vector spaces.

Our departure point is a mixed tensor $\gamma_{aB}{}^C$ fulfilling the algebraic property

$$\gamma_{aA}{}^B\gamma_{bB}{}^C + \gamma_{bA}{}^B\gamma_{aB}{}^C = -\delta_A{}^C g_{ab}, \tag{1}$$

where $\delta_A{}^C$ is the identity tensor (also known as the *Kronecker delta*) on the vector space $\mathbf{S}$ (note that each tensorial quantity has a number of slots arranged sequentially on which the indices are filled). Eq. (1) means that $\gamma_{aA}{}^B$ belongs to a representation on the vector space $\mathbf{S}$ of the *Clifford algebra* $Cl(\mathbf{L}, g)$. When this representation turns out to be irreducible then there are strong constraints on the vector space $\mathbf{S}$ which we summarise in the following result.

Theorem 1. *If the quantity $\gamma_{aB}{}^C$ belongs to an irreducible representation of $Cl(\mathbf{L}, g)$, then the dimension of $\mathbf{S}$ is 4 and there exist two antisymmetric spinors ϵ_{AB}, $\widehat{\epsilon}^{AB}$, unique up to a constant, such that*

$$\epsilon_{AB}\widehat{\epsilon}^{CB} = \delta_A{}^C. \tag{2}$$

These antisymmetric spinors are related to $\gamma_{aB}{}^C$ by means of the algebraic property

$$\gamma_{aD}{}^A\gamma^a{}_C{}^B = \frac{1}{2}\delta_D{}^A\delta_C{}^B - \delta_C{}^A\delta_D{}^B + \epsilon_{CD}\widehat{\epsilon}^{AB}. \tag{3}$$

This theorem is a particular case of a more general result described in appendix B of [8]. Another important consequence of the irreducibility of the Clifford algebra representation is that the quantity $\gamma_{aA}{}^A$ must vanish, for otherwise the 1-form $\gamma_{aA}{}^A$ would be invariant under the action of any endomorphism of $\mathbf{L}$ keeping g_{ab} invariant (orthogonal group), and this can only happen for scalars and 5-forms [8].

Previous ideas lead naturally to the concept of *spin structure.*

Definition 1. *Under the conditions stated in theorem 1 we will refer to $\gamma_{aA}{}^B$ as a spin structure on $\mathbf{L}$. The complex vector space $\mathbf{S}$ is then called the spin space of the spin structure.*

The spinors ϵ_{AB} and $\widehat{\epsilon}^{AB}$ can be regarded as a metric tensor and its inverse in the vector space $\mathbf{S}$ and therefore they can be used to raise and lower spinorial indices. Since they are antisymmetric quantities we must pay close attention to the conventions introduced for these operations, which in our case are

$$\xi^A\epsilon_{AB} = \xi_B\ , \quad \xi^A = \widehat{\epsilon}^{AB}\xi_B.$$

In particular we can raise the indices of ϵ_{AB} getting $\epsilon^{AB} = \widehat{\epsilon}^{AB}$ and from now on, only the symbol ϵ will be used for the symplectic metric and its inverse. Note also the property

$$\delta^A{}_B = -\delta_B{}^A. \tag{4}$$

Here the quantity $\delta_B{}^A$ is the *Kronecker delta* on $\mathbf{S}$ and $\delta^A{}_B$ is a derived quantity obtained from it by the raising and lowering of indices It is possible to use the quantity $\gamma^a{}_{AB}$ to relate spinors to tensors and back. For example for a vector v^a, its spinor counterpart is given by

$$v^{AB} = \gamma_a{}^{AB}v^a.$$

A consequence of (3) is that $\gamma^a{}_{[AB]} = \gamma^a{}_{AB}$ which means that v^{AB} is anti-symmetric and also traceless, because $\gamma_{aA}{}^A = 0$. Reciprocally, any antisymmetric and traceless spinor ξ^{AB} has a unique vector counterpart given by $\xi^{AB}\gamma^a{}_{AB}$. It is straightforward to generalise this simple example to spinors and tensors of higher rank (see [4] for the precise statement).

3. Spinor calculus

In this section we explain how the algebraic considerations discussed before are translated to a 5-dimensional Lorentzian manifold This requires the introduction of the concepts of *spin bundle* and *spin structure*. Once these are defined we will go on to show how a calculus involving spinors is developed.

3.1. Spin structures on a 5-dimensional Lorentzian manifold

Suppose that $(\mathcal{M}, g)$ is a 5-dimensional Lorentzian manifold and let $T_p(\mathcal{M})$ be the tangent space at a point p. This is a vector space which can be endowed with the Lorentzian scalar product $g(\ ,\ )|_p$. Therefore the vector space $T_p(\mathcal{M})$ has properties similar to $\mathbf{L}$ and we can introduce a spin space $\mathbf{S}_p$ and a spin structure ${\gamma_{aA}}^B|_p$ at each point p.

Definition 2 (**Spin bundle**). *The union*

$$S(\mathcal{M}) \equiv \bigcup_{p\in\mathcal{M}} \mathbf{S}_p, \tag{5}$$

is a vector bundle with the manifold $\mathcal{M}$ as the base space and the group of linear transformations on $\mathbb{C}^4$ as the structure group. We will call this vector bundle the spin bundle and the sections of $S(\mathcal{M})$ are the contravariant rank-1 spinor fields on $\mathcal{M}$.

One can now formulate a similar definition starting from spin spaces of spinors of arbitrary rank and we will use the name "spin bundle" for any of the vector bundles constructed in this way. Also it is possible to construct new vector bundles by combining spin bundles and vector bundles of tensors, using the tensor product operation. We will use the symbol $\mathfrak{S}(\mathcal{M})$ to denote generically any of these vector bundles.

Definition 3 (**Spin structure on a 5-dimensional manifold**). *If the quantity ${\gamma_{aA}}^B|_p$ varies smoothly on the manifold $\mathcal{M}$, then one can define a smooth section on the bundle $\mathfrak{S}(\mathcal{M})$, denoted by ${\gamma_{aA}}^B$. We call this smooth section a smooth spin structure on the Lorentzian manifold $(\mathcal{M}, g)$.*

A necessary and sufficient condition for the existence of a spin structure is that the second Stiefel-Whitney class of $\mathcal{M}$ vanishes (see e.g. [5]).

3.2. Spin covariant derivative

One can introduce a covariant derivative in the bundle $\mathfrak{S}(\mathcal{M})$ by following the standard axioms which define a covariant derivative in any vector bundle.

Definition 4 (**Spin covariant derivative**). *Suppose that $\mathfrak{S}(\mathcal{M})$ admits a spin structure ${\gamma_{aA}}^B$. We say that a torsion-free covariant derivative D_a defined on $\mathfrak{S}(\mathcal{M})$ is compatible with the spin structure ${\gamma_{aA}}^B$ if it fulfils the property*

$$D_a {\gamma_{bC}}^D = 0. \tag{6}$$

The covariant derivative D_a is then called a spin covariant derivative with respect to the spin structure ${\gamma_{aA}}^B$.

Elementary properties of a spin covariant derivative are $D_a g_{bc} = 0$ and $D_a \epsilon_{AB} = \epsilon_{AB} D_a Y$, for some scalar function Y. The first property means that D_a is just the Levi-Civita covariant derivative when acting on tensors. From these properties is clear that a spin covariant derivative with respect to a given spin structure is not unique unless additional restrictions are imposed. The next result, proven in [4] spells out which these restrictions are.

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Theorem 2. *There is one and only one spin covariant derivative ∇_a on $\mathfrak{S}(\mathcal{M})$ with respect to the spin structure $\gamma_{aA}{}^{B}$ which fulfils the property*

$$\nabla_a \epsilon_{AB} = 0. \tag{7}$$

From now on we will work with the spin covariant derivative described in theorem 2. Some spinor equations adopt a simpler form when written in terms of the differential operator $\nabla_{AB} \equiv \gamma^a{}_{AB} \nabla_a$.

3.3. The curvature spinors

The curvature of ∇_a has two contributions: one coming from its role as a connection in the spin bundle and another from its role as a connection in the tensor bundle. The latter is just the Riemann tensor of the Levi-Civita covariant derivative whereas the former is characterised by a certain set of *irreducible* spinors called *curvature spinors*. The Riemann tensor and the curvature spinors are related as shown in the following theorem, proven in [4].

Theorem 3. *Define the quantity $G^{ab}{}_{AC} \equiv -\gamma^a{}_{(A}{}^{B} \gamma^b{}_{C)B}$. Then the Riemann tensor R_{abcf} of the covariant derivative ∇_a can be decomposed in the form*

$$R_{abcf} = \Lambda(g_{af} g_{bc} - g_{ac} g_{bf}) - \frac{1}{2} G_{ab}{}^{AB} G_{cf}{}^{CD} \Psi_{ABCD} - G_{ab}{}^{AB} G_{cf}{}^{CD} \Omega_{ACBD}. \tag{8}$$

The quantities Λ, Ω_{ABCD} and Ψ_{ABCD} are known as the <u>curvature spinors</u>. *They fulfil the algebraic properties*

$$\Psi_{(ABCD)} = \Psi_{ABCD}\ , \quad \Omega_{ABCD} = \Omega_{[AB]CD} = \Omega_{CDAB}\ , \quad \Omega_{AB}{}^{C}{}_{C} = \Omega_{A}{}^{C}{}_{CD} = 0\ ,$$
$$\Omega_{ABCD} + \Omega_{BCAD} + \Omega_{CABD} = 0\ , \tag{9}$$

which correspond to the first Bianchi identity of R_{abcd}, and the differential identity

$$\nabla_{(Z}{}^{W} \Psi_{V)BAW} - \nabla_{(Z}{}^{W} \Omega_{V)ABW} - \nabla_{(Z}{}^{W} \Omega_{V)BAW} - 2\epsilon_{(A|(V} \nabla_{Z)|B)} \Lambda = 0\ , \tag{10}$$

which corresponds to the second Bianchi identity of R_{abcd}.

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